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Cumulative nonlinear distortion of an acoustic wave propagating through non-uniform flow

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In this paper we examine how the unsteady flow field radiated from an oscillating body is altered from the result of acoustic theory as the direct consequence of disturbances propagating through the non-uniform flow produced by the presence of the body. Taking the specific example of an oscillating airfoil placed in supersonic flow and having the contour of a parabolic arc, we derive a closed-form representation for the unsteady flow field in terms of the confluent hypergeometric function. The analytical expression reveals explicitly that, though the body shape has a negligible effect in the near field, it inextricably affects the unsteady flow at a large distance, both in its amplitude and phase, and substantially modifies the results of acoustic theory. In addition, we display the relation of this solution to the 'fundamental solution' and the other salient physical features connected with disturbances propagating through non-uniform flow. The present results recover Whitham's rule in the limit of zero frequency of oscillation and also include, as another special case, the unsteady solution for a wedge obtained by Carrier and Van Dyke.

1. Introduction

As is well known, acoustic theory in a moving medium is based on two major assumptions: that a disturbance propagates at a *uniform* acoustic velocity and is swept downstream at a *constant* free-stream speed. Although this approximation is sufficiently accurate in the vicinity of the body, the acoustic theory for a supersonic flow is manifestly unfit for the description of the far field; it fails, for example, to reproduce the fanning out or coalescence of Mach waves. The reasons for the breakdown have long been understood (e.g. Lighthill 1954): as a wavelet spreads out, two nonlinear effects ignored in the acoustic theory, i.e. the non-uniform acoustic and flow velocities, which vary with both position and time, emerge and exert an influence over a large distance. The nonlinear effects are locally small everywhere, including the far field. However, not only is the disturbance at a given point influenced by the slightly perturbed flow properties at that location but it has been undergoing a continual distortion while propagating through a non-uniform flow field. It is this cumulative distortion or 'memory' content of the signal which encroaches upon the result of acoustic theory and eventually alters it in the far field.

For a steady flow, the task of surmounting the shortcomings of acoustic theory has drawn the attention of Friedrichs (1948), Lighthill (1949) and Whitham (1950, 1952), to mention only a few. These efforts culminated in the following celebrated rule due

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to Whitham† (1952): to a good approximation, the result of acoustic theory can be amended if one replaces the linearized Mach wave by one revised using linearized velocities but, along this improved Mach wave, retains the values of the fluid properties predicted by acoustic theory. Crudely speaking, then, the only visible consequence of the nonlinearity is the directional change in the Mach waves; the fluid velocities remain essentially unchanged. We reiterate here that the flow is steady in the frame of reference fixed to the body.

In contrast to the above steady flow situation, relatively less attention appears to have been paid to problems where the flow is unsteady, again with respect to the co-ordinate system fixed to the body. To be sure, related studies have been published but they seem mostly to be restricted to a one-dimensional problem and its diverse variants (e.g. Lesser 1970; Romanova 1970; Nayfeh 1975). There have been very few attempts, if any, to obtain, in the spirit of the above steady problems, a complete and uniformly valid solution and then display the global behaviour of the unsteady flow field in either two- or three-dimensional space. Yet there are many important practical problems, like the unsteady aerodynamic interference between a multitude of oscillating bodies in a flow, e.g. flutter of cascaded airfoils, and other similar phenomena, where such an improved prediction of the unsteady flow valid even in the far field is critically needed. Prompted by this, we address here the problem of obtaining a first-order correction to the acoustic field radiated from an oscillating body, accounting for the interaction with the non-uniform flow created by the body itself.

In the case of unsteady flow, the nonlinearity will have additional consequences, as one can anticipate from the following physical reasoning. Let us first assume that only a single point on the body is oscillating sinusoidally. When one plots at a given point in the flow the time trace of the disturbance emitted, the departure of the non-uniform acoustic and convective velocities from the uniform ones (from acoustic theory) will be graphically revealed, mainly, as a phase difference between the actual signal and the one predicted by the acoustic theory. The phase lag depends on the position and, the more one moves away from the source, the more the phase lag will increase. Suppose now that the whole body is oscillating. Then the above phase lag for an individual disturbance, which differs from one signal to another, and that alone (to say nothing of the modification in the amplitude of each signal) could introduce, *when signals are vectorially added*, a pronounced correction to both the *amplitude* and the *phase* of the unsteady flow in the far field. Thus the nonlinearity would cause, in addition to the alteration to be made to the direction of the characteristics, a change in the fluctuating pressure itself. The modification induced in the far-field signal has the following implication, which appears to warrant emphasis: contrary to the situation in the near field, the unsteady signal at a large distance, even to first order in the small perturbation, can by no means be separated from such effects as the body shape, camber and angle of attack, which cause the properties of propagation to be non-uniform. The effect of thickness, for example, would be inextricably embedded in the far-field unsteady signal.

Our present aim is to confirm these expectations and we shall do so by investigating

† In early literature this was referred to as Whitham's hypothesis. Now that it has become well established, it appears more appropriate to call it a rule instead. This rule should not, of course, be confused with another rule, due also to Whitham, relevant to the propagation of a shock through a region of varying cross-sectional area (e.g. Whitham 1974).

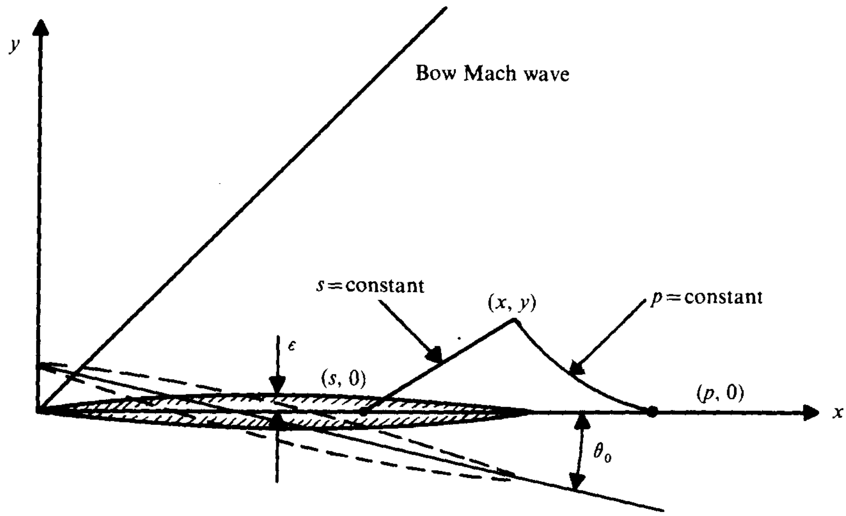


FIGURE 1. Definition sketch.

the effects of a non-planar body, whose presence creates a non-uniform surrounding environment, upon the unsteady flow field. We shall expressly limit our investigations to the case of a two-dimensional slender body whose upper and lower surfaces consist of parabolic convex arcs and which is oscillating sinusoidally in a supersonic flow (figure 1). Though, by confining our attention to this particular shape, we shall inevitably forfeit formal generality, the present approach will give a closed-form solution which is amenable to detailed study; from this we hope to glean the essential features of the flow non-uniformities. With regard to figure 1 again, the thickness of the body is characterized by a parameter ϵ and the amplitude of oscillation by θ_0. We shall examine the cumulative effects of the second-order terms, which ascend in the far field to a first-order unsteady term $O(\theta_0)$. There are three second-order terms, $O(\epsilon\theta_0)$, $O(\theta_0^2)$ and $O(\epsilon^2)$, of which only the first two are relevant for the present unsteady problem. If one assumes $\epsilon \gg \theta_0$, one can discard the term $O(\theta_0^2)$, whose presence would cause undesirable higher harmonics. With this assumption, we are now in a position to focus attention on the remaining, $O(\theta_0\epsilon)$ term, which represents the genuine coupling effect of present interest. It should be remembered, however, that, as pointed out by Hayes (1954) for steady flow, only a few selected second-order terms contribute cumulatively to the first-order effects. Hence we shall pick out, by the use of the strained co-ordinate technique, those terms $O(\theta_0\epsilon)$ whose cumulative effects amount to $O(\theta_0)$ in the far field. Thus our aim is clearly different from Van Dyke's (1953*a*) second-order theory for an oscillating airfoil including the effect of thickness. There, because of his interest in the flow on the airfoil surface, combined with a situation involving only slow oscillations, he used a regular perturbation scheme in θ_0 and ϵ and obtained a solution to $O(\theta_0\epsilon)$; consequently, the hierarchical ascent of terms of second order to first order in the far field was neither expected to take place nor was his concern. On the contrary, our interest centres on just such an evolutionary, ascending process.

In the next section we shall begin with the governing equation and simplify it in §3 by employing the strained co-ordinate technique. In §§4 and 5, we shall describe the procedure for solving this simplified equation. We set out to obtain the corresponding Riemann function appropriate for a parabolic airfoil; the Riemann function can be constructed explicitly and exactly in terms of the confluent hypergeometric function. With the Riemann function thus obtained, the solution, equation (5.4), follows from it without much difficulty. In §6, before embarking on a physical interpretation of the solution, we pause and confirm that the present results can be reduced, through the limiting properties of the confluent hypergeometric function, to some known results. In the limit of zero frequency of oscillation, we shall recover Whitham's rule; for an oscillating wedge with small apex angle, the present result will embrace, as a special case, the solution obtained by Carrier (1949) and Van Dyke (1953*b*). We shall resume the discussion of the curved airfoil in §7, where we observe that Tricomi's (1949) expansion formula for the confluent hypergeometric function is ideally suited to the extraction of a physical interpretation; the gradual ascent of second-order terms to alter the acoustic signal in the far field will become effortlessly visible; and there the effect of body shape will be found to be tenaciously inseparable from the unsteady flow field. This will be followed in §8 by further description of salient physical features related to the disturbances propagating through the non-uniform medium.

2. Problem formulation

The governing equation for the perturbed velocity potential Φ is given to second order, to which order the flow can still be regarded to be irrotational, by

$$\begin{aligned}\Phi_{yy}-m^2\Phi_{xx}-2\frac{U_\infty}{a_\infty^2}\Phi_{xt}-\frac{1}{a_\infty^2}\Phi_{tt} &= \frac{M_\infty^2}{U_\infty}\left\{(\gamma-1)\left(\Phi_x+\frac{1}{U_\infty}\Phi_t\right)(\Phi_{xx}+\Phi_{yy})+2\Phi_x\Phi_{xx}+2\Phi_y\Phi_{xy}\right.\\ &\qquad\left.+\frac{2}{U_\infty}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\right\}, \qquad (2.1)\end{aligned}$$

where the perturbed velocity components (u',v') are related to Φ by

$$\Phi_x = u', \quad \Phi_y = v',$$

U_∞ is the free-stream velocity, a_∞ the speed of sound in the free stream, $M_\infty = U_\infty/a_\infty$, $m=(M_\infty^2-1)^{\frac{1}{2}}$ and γ is the adiabatic exponent of the gas. We express, according to Van Dyke (1953*a*), the co-ordinate of the moving upper surface as

$$y = \epsilon f(x)-\theta_0 e^{i\omega t}g(x), \qquad (2.2)$$

where $\epsilon f(x)$ $(\epsilon \ll 1)$ designates the shape of the body in its mean position of oscillation and the second term represents its harmonic motion with frequency ω. The two small non-dimensional parameters ϵ and θ_0 characterize the slenderness of the body and the amplitude of motion, respectively. As long as the shock remains attached, we need consider only the flow above the upper surface. The boundary condition on the surface of the airfoil, as given by Van Dyke (1953*a*) to second order, is

$$\Phi_y = (U_\infty+\Phi_x)(\epsilon f'-\theta_0 e^{i\omega t}g')-i\omega\theta_0 e^{i\omega t}g-(\epsilon f-\theta_0 e^{i\omega t}g)\Phi_{yy} \quad \text{at} \quad y=0. \qquad (2.3)$$

Also, Φ vanishes upstream of the bow shock, whose position moves in time. Since the flow variables are discontinuous at the shock and, strictly speaking, do not possess

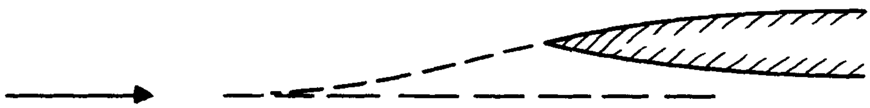


FIGURE 2. Smoothing technique of Van Dyke.

derivatives there, the governing equation is not formally satisfied. Hence in principle jump conditions across the shock, which is moving and whose temporal position is unknown *a priori*, must be imposed to ensure the conservation of mass, momentum and energy there; this would introduce complications. However, this knotty problem can be completely circumvented by the smoothing technique, which was first devised by Courant & Friedrichs (1948, p. 365) for steady flow and later extended to the unsteady case by Van Dyke (1953*a*). We first imagine that an extension has been added to the leading edge of the actual airfoil: a sufficiently smooth and flexible tip of such a shape and moving in such a way as to prevent the formation of the shock in the flow above the upper surface (figure 2). We then regard the desired solution as the limit as the extension shrinks. Once this device has been employed, as here, the need to impose jump conditions at the shock can be eliminated for the solutions up to second order. (Also, whenever necessary, we shall hereafter regard the discontinuity in the flow variables at the shock in the sense of the above limiting process.) The smoothing technique provides, in effect, a formal justification for the following point of view: the global behaviour of the unsteady flow downstream of the bow shock can, to a good approximation, be determined essentially independently of the presence of the shock and various complications arising from its motion (except in the close vicinity of the shock, where such a solution fails); the situation is akin to the familiar steady problem (Whitham 1952).

Following Van Dyke a little further, we separate the perturbed velocity potential into a steady and time-dependent part by writing

$$\Phi = \epsilon\phi(x,y:\epsilon) + \theta_0 \exp\left[i(\omega t - kx)\right]\psi(x,y:\epsilon,\theta_0), \tag{2.4}$$

where $k = M_\infty^2\omega/m^2U_\infty$. The first term represents the steady base flow and ψ in the second term corresponds to the unsteady flow; our interest is focused on ψ. We substitute (2.4) into (2.1) and (2.3) and assume $\epsilon \gg \theta_0$, as stated in the introduction. We thus obtain the following two sets of equations: for ϕ

$$\epsilon[-m^2\phi_{xx} + \phi_{yy}] = \epsilon^2(M_\infty^2/U_\infty)[m^2(N-1)\phi_x^2 + \phi_y^2]_x, \tag{2.5a}$$

with the boundary condition

$$\epsilon\phi_y = \epsilon U_\infty f' + \epsilon^2(\phi_x f' - f\phi_{yy}) \quad \text{at} \quad y = 0, \tag{2.5b}$$

and for ψ

$$\begin{aligned}\theta_0[-m^2\psi_{xx} + \psi_{yy} - (km/M)^2\psi] = {} & 2(\epsilon\theta_0/U_\infty)\{M_\infty^2[m^2(N-1)\phi_x\psi_x + \phi_y\psi_y]_x \\ & -ik[(2N-1)m^2\phi_x\psi_x + Nm^2\psi\phi_{xx} + \phi_x\psi_y] - N(km/M_\infty)^2\phi_x\psi\},\end{aligned} \tag{2.6a}$$

where $N = \frac{1}{2}(\gamma+1)M_\infty^2/m^2$, with the boundary condition

$$\begin{aligned}\theta_0\psi_y = {} & -\theta_0[U_\infty g' + (ikU_\infty m^2/M_\infty^2)g]e^{ikx} \\ & + \epsilon\theta_0[f'(\psi_x - ik\psi) + (-\phi_x g' + g\phi_{yy})e^{ikx} - f\psi_{yy}] \quad \text{at} \quad y = 0.\end{aligned} \tag{2.6b}$$

In obtaining the above equation, some simplification on the right-hand side has been achieved by using the expressions for the left-hand side and neglecting terms of higher than the second order. It should be noticed that, although (2.5a) is nonlinear in ϕ, equation (2.6a), the basis of this paper, is a *linear*† function of ψ involving variable coefficients. Both ϕ and ψ vanish upstream of the bow shock.

3. Application of strained co-ordinate technique

The right-hand sides of (2.5a) and (2.6a) are of higher order than the left-hand sides. Consequently, if one uses a regular perturbation scheme, they successively yield the first- and second-order equations of Van Dyke with the right-hand sides either zero or expressible in terms of the first-order velocities, respectively; the first-order equation for ψ, in particular, is the (reduced) acoustic equation and it obviously precludes the ascent of the terms on the right-hand side to first order. In order to achieve our stated objective of examining such an evolutionary process, we shall employ the strained co-ordinate technique instead: this is the point of departure of the present analysis. Although the original strained co-ordinate technique developed by Lighthill and Whitham involves only a single family of characteristics, the present unsteady problem requires two families of characteristics for adequate description of the flow field. It is therefore convenient to use Lin's (1954) extension of the strained co-ordinate technique (see also Oswatitsch 1962) or the analytic method of characteristics, which enables one to treat the case of two families of characteristics. According to this method, the independent variables (x, y) as well as the dependent variables are to be expanded, with the characteristic parameters s and p regarded as new independent variables:

$$x = x^{(0)}(s,p) + \epsilon x^{(1)}(s,p) + \dots, \tag{3.1a}$$

$$y = y^{(0)}(s,p) + \epsilon y^{(1)}(s,p) + \dots, \tag{3.1b}$$

$$\epsilon\phi = \epsilon\phi^{(1)}(s,p) + \epsilon^2\phi^{(2)}(s,p) + \dots, \tag{3.2a}$$

$$\theta_0\psi = \theta_0\psi^{(1)}(s,p) + \epsilon\theta_0\psi^{(2)}(s,p) + \theta_0^2\psi^{(3)}(s,p) + \dots, \tag{3.2b}$$

where s and p are constant along the corresponding characteristic curves, respectively. With respect to the characteristic curve, we first observe that, comparing (2.5a) and (2.6a), all the coefficients of the second derivatives in the steady equation are the same as the corresponding ones in the unsteady part. This dictates, then, that the characteristic curves for both the steady and the unsteady equations are identically the same and given by

$$\left.\frac{dy}{dx}\right|_{s=\text{const}} = \frac{1}{m}\left[1 - \epsilon\frac{M_\infty^2}{U_\infty}(N-1)\phi_x + \epsilon\frac{M_\infty^2}{U_\infty}\frac{1}{m}\phi_y\right] + \dots, \tag{3.3a}$$

$$\left.\frac{dy}{dx}\right|_{p=\text{const}} = -\frac{1}{m}\left[1 - \epsilon\frac{M_\infty^2}{U_\infty}(N-1)\phi_x - \epsilon\frac{M_\infty^2}{U_\infty}\frac{1}{m}\phi_y\right] + \dots. \tag{3.3b}$$

† Besides the usual shock emanating from the leading edge (and the one at the trailing edge, which does not matter for the flow field upstream of it), no additional shock is created owing to the motion of the airfoil; consequently, the entire unsteady flow can be uniformly described by the linearized equation.

Into these we substitute (3.1 a, b) and (3.2 a) and equate the coefficients of equal powers of ϵ. We then determine successively, using the boundary condition (2.5 b), the terms in the series expansion; while the zeroth-order terms in (3.1), $x^{(0)}$ and $y^{(0)}$, give the expression for the characteristic parameters corresponding to acoustic theory, the first-order terms, $x^{(1)}$ and $y^{(1)}$, give the desired nonlinear correction. We direct attention towards the fact that the procedure is dependent wholly on the steady flow and excludes the unsteady part (3.2 b). This process of co-ordinate stretching in steady flow being familiar, it suffices here to write down the following results:

$$\phi^{(1)} = -H(s)(U_\infty/m)f(s), \tag{3.4}$$

$$s = x - my - \epsilon myN(M_\infty^2/U_\infty)\,d\phi^{(1)}(s)/ds, \tag{3.5a}$$

$$p = x + my - (\epsilon/2U_\infty)(N-2)M_\infty^2[\phi^{(1)}(s) - \phi^{(1)}(p)], \tag{3.5b}$$

where $H(s)$ is a unit step function. The above expressions for s and p have been put in the present form by rewriting the results corresponding to (3.1 a, b). Geometrically, s represents, as shown in figure 1, the root of the straight Mach wave passing through a given point (x, y) and along this s remains constant (Van Dyke 1975); likewise, p represents the root of the cross Mach wave, along which p remains constant. (As a matter of fact, the constants of integration in (3.3), their choice being at our disposal, are so adjusted that, at $y = 0, x = s = p$.) Equation (3.4) indicates that the steady, first-order velocity potential is dependent on s only and it obviously embodies Whitham's rule.

Having thus specified s and p, we then substitute the expansion for the unsteady part (3.2 b) into (2.6). In obtaining the equation for the leading term $\psi^{(1)}$, we proceed with caution and retain the terms associated with k on the right-hand side because, for sufficiently high frequencies, they could become comparable with the terms on the left-hand side; the terms not associated with k can be neglected. One thus obtains

$$\theta_0\left\{-4\psi_{sp}^{(1)} + \frac{4i}{U_\infty}N\epsilon k\phi'^{(1)}\psi_s^{(1)} + \frac{4i}{U_\infty}(N-1)\epsilon k\phi'^{(1)}\psi_p^{(1)} + \left[-\left(\frac{k}{M_\infty}\right)^2 + \frac{2i}{U_\infty}N\epsilon k\phi''^{(1)}\right]\psi^{(1)}\right\} = 0, \tag{3.6a}$$

where $\phi'^{(1)}$ designates the derivative of $\phi^{(1)}$ with respect to s; in differentiating $\phi^{(1)}$, we recall and envisage the smoothing process described in §2 and discard the term associated with the delta function. (When obtaining (3.6 a), the term $(k/M_\infty)^2$ in the braces initially appears as $(k/M_\infty)^2[1-2(N/U_\infty)\epsilon\phi'^{(1)}]$ but the second term in the square brackets is neglected.) The boundary condition (2.6 b) becomes

$$\theta_0\{i\epsilon kf'(s)\psi^{(1)} - m\psi_s^{(1)} + m\psi_p^{(1)} + V(s)\,e^{iks}\} = 0 \quad \text{at} \quad s = p,$$

where

$$V(x) = U_\infty g'(x) + (ikU_\infty m^2/M_\infty^2)g(x). \tag{3.6b}$$

Also, the upstream condition becomes

$$\psi^{(1)} = 0 \quad \text{for} \quad s < 0. \tag{3.6c}$$

It is convenient at this point to introduce the function F defined by

$$\psi^{(1)} = \exp[i(\epsilon k/U_\infty)N(p-s)\phi'^{(1)}(s)]F. \tag{3.7}$$

Then (3.6a) becomes, to the order consistent with the present approximation,

$$\theta_0\left\{F_{sp}+\frac{i\epsilon k}{U_\infty}[-(2N-1)\phi'^{(1)}(s)+(p-s)N\phi''^{(1)}(s)]F_p\right.$$
$$\left.+\left[\left(\frac{k}{2M_\infty}\right)^2+\frac{i\epsilon k}{2U_\infty}N\phi''^{(1)}(s)\right]F\right\}=0, \quad (3.8a)$$

with the boundary conditions

$$\theta_0\left\{i\epsilon k\left[-\frac{1}{m}f'(s)-\frac{2}{U_\infty}N\phi'^{(1)}(s)\right]F+F_s-F_p-\frac{1}{m}V(s)\,e^{iks}\right\}=0 \quad \text{at} \quad s=p, \quad (3.8b)$$

and

$$F=0 \quad \text{for} \quad s<0. \quad (3.8c)$$

In (3.8a,b), the factor θ_0 is retained as a reminder that the equations are valid to order θ_0, the higher-order terms such as those $O(\epsilon\theta_0)$ in (3.2b) being neglected. Our aim is to obtain the explicit solution for F and we shall do so for an airfoil whose shape consists of parabolic arcs.

4. Construction of the Riemann function

If $\phi'^{(1)}(s)$ were either zero or a constant, (3.8a) would be reduced to the telegraph equation. In the present case of a parabolic-arc airfoil, $f(x)$ in (2.2) is quadratic in x and from (3.4) the derivative $\phi'^{(1)}$ is linear in s. Thus (3.8a) is a second-order linear hyperbolic equation whose coefficients are variable (and linear in s). It is well known that the solution of any second-order linear hyperbolic equation can be expressed in the form of an integral representation, once the corresponding Riemann function has been obtained (e.g. Courant & Hilbert 1962, p. 449). If, in general, u satisfies

$$\mathscr{L}[u]\equiv u_{xy}+au_x+bu_y+cu=0,$$

where a, b and c are given functions of x and y, then u can be represented by an integral along the boundary (where Cauchy data are assumed to be prescribed) whose integrand involves the Riemann function R of the operator $\mathscr{L}$. R does not satisfy the operator equation $\mathscr{L}(R)=0$ but rather satisfies the adjoint operator equation

$$\mathscr{L}^*[R]\equiv R_{xy}-(aR)_x-(bR)_y+cR=0.$$

For our purpose, it is convenient to derive first, instead of R, the Riemann function R^* of the adjoint operator which satisfies the operator equation for $\mathscr{L}$ itself; then we obtain R through the symmetry property of the Riemann functions. For the present equation (3.8a), the Riemann function of the adjoint operator $R^*(\xi,\eta;s,p)$ satisfies the following three conditions (Courant & Hilbert, *ibid.*):

(*a*) $$\mathscr{L}_{\xi,\eta}[R^*]=R^*_{\xi,\eta}+\frac{i}{U_\infty}\epsilon k[-(2N-1)\phi'^{(1)}(\xi)+(\eta-\xi)N\phi''^{(1)}(\xi)]R^*_\eta$$
$$+\left[\left(\frac{k}{2M_\infty}\right)^2+\frac{iN}{2U_\infty}\epsilon k\phi''^{(1)}(\xi)\right]R^*=0; \quad (4.1a)$$

(*b*) along AC in figure 3,

$$\frac{1}{R^*}\frac{\partial R^*}{\partial\eta}=0 \quad \text{on} \quad \xi=s, \quad (4.1b)$$

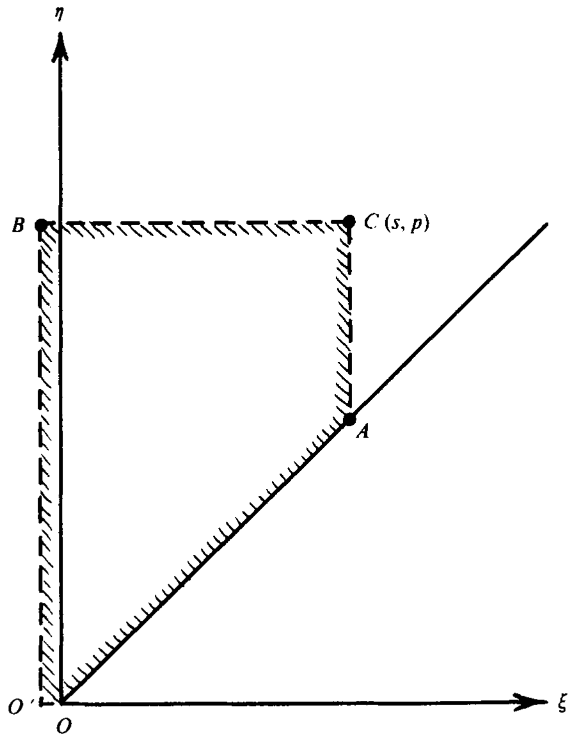


FIGURE 3. Integration contour.

and along BC,

$$\frac{1}{R^*}\frac{\partial R^*}{\partial \xi} = \frac{i}{U_\infty}\epsilon k[(2N-1)\phi'^{(1)}(\xi) - (\eta-\xi)N\phi''^{(1)}(\xi)] \quad \text{on} \quad \eta = p; \tag{4.1c}$$

(c) $R^*(s,p;s,p) = 1.$ (4.1 d)

Integrating (4.1 c) and determining the constant of integration from (4.1 d), we obtain

$$R^*(\xi,p;s,p) = \exp \mu^*, \tag{4.2}$$

where

$$\mu^* = (i/U_\infty)\epsilon k\{(N-1)[\phi^{(1)}(\xi) - \phi^{(1)}(s)] - N[p(\phi'^{(1)}(\xi) - \phi'^{(1)}(s)) - (\xi\phi'^{(1)}(\xi) - s\phi'^{(1)}(s))]\}.$$

If we write

$$R^*(\xi,\eta;s,p) = \exp(\mu^*)\,M(z), \tag{4.3}$$

where

$$z = -(i/U_\infty)N\epsilon k(\xi - s)(\eta - p)\phi''^{(1)}, \tag{4.4}$$

then for a parabolic-arc airfoil, for which $\phi''^{(1)}$ is a constant, (4.1 a) is reduced to the following ordinary differential equation:

$$zM'' + (1-z)M' - aM = 0, \tag{4.5}$$

where

$$a = \tfrac{1}{2} + U_\infty k(4i\epsilon N M_\infty^2 \phi''^{(1)})^{-1}.$$

This is known as Kummer's equation and its only solution which satisfies (4.1 d) is the following confluent hypergeometric function (e.g. Slater 1960, p. 2):

$$M \equiv M(a, 1, z), \tag{4.6}$$

defined by

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{a_2 z^2}{b_2 2!} + \ldots + \frac{a_n z^n}{b_n n!} + \ldots,$$

where

$$a_n = a(a+1)(a+2)\ldots(a+n-1), \quad \text{for } n = 1, 2, \ldots,$$

and

$$a_0 = 1.$$

Hence (4.3) becomes

$$R^*(\xi, \eta; s, p) = \exp(\mu^*) M(a, 1, z). \tag{4.7}$$

Along $\xi = s$, $R^*(s, \eta; s, p) = 1$ and this obviously satisfies the remaining requirement (4.1 b) for the Riemann function.

The Riemann function $R(\xi, \eta; s, p)$ may be immediately derived from R^* through the symmetry property of the Riemann function (Courant & Hilbert 1962, p. 454) by replacing ξ and η with s and p, respectively. Thus we obtain

$$R(\xi, \eta; s, p) = \exp(\mu) M(a, 1, z), \tag{4.8}$$

where

$$\mu = (i/U_\infty)\epsilon k\{(N-1)[\phi^{(1)}(s) - \phi^{(1)}(\xi)] - N[\eta(\phi'^{(1)}(s) - \phi'^{(1)}(\xi)) - (s\phi'^{(1)}(s) - \xi\phi'^{(1)}(\xi))]\},$$

$$a = \tfrac{1}{2} + U_\infty k(4i\epsilon N M_\infty^2 \phi''^{(1)})^{-1}$$

and

$$z = -(i/U_\infty)\epsilon k N(s-\xi)(p-\eta)\phi''^{(1)}.$$

5. An integral representation of the solution

Once the Riemann function has been thus derived, one is in a position to employ Riemann's formula (Courant & Hilbert, *ibid.*) to obtain the integral representation of F in (3.8 a), provided that Cauchy data are prescribed on the boundary. Unfortunately, the present boundary condition (3.8 b), which applies along the segment OA of figure 3 (this corresponds to the x axis of figure 1), is not Cauchy data. Rather, it expresses a linear relationship between the function F and its derivatives; this induces some complication. If one applies Riemann's formula to the contour around the shaded region of figure 3 ($OACBO'$), although the contributions from the line segments AC, CB, BO' and OO' vanish identically, one ends up with an integral along OA; since it turns out that the integral involves the value of F, which is unknown as yet, one has to solve a complicated integral equation to determine it.

The difficulty is by no means unique to the parabolic airfoil, and in fact the same complication arises even in the more simplified situation of a flat-plate airfoil, where $\phi'^{(1)}$ is zero. In such a case, (3.8 a) is reduced to the telegraph equation and the corresponding Riemann function is a Bessel function (e.g. Courant & Hilbert, *ibid.*):

$$R(\xi, \eta; s, p) = J_0\{(k/M_\infty)[(s-\xi)(p-\eta)]^{\frac{1}{2}}\}. \tag{5.1}$$

To construct the flat-plate solution, Temple & Jahn (1945) used this and applied Riemann's formula for a closed curve; the contour around the shaded region of figure 3 is none other than their path of integration. Their final result for F at a general point in the flow was left in a somewhat awkward form involving, inside the integral along the segment OA, the unknown values of F to be evaluated there, although the undesirable term vanishes from the integral for a point on the surface of airfoil, i.e. on OA. It turns out, however, that one can advance a step further and eliminate the term entirely. More specifically, by substituting the expression derived for F on the surface of the airfoil into the integral representation for an arbitrary point and noting an identity involving a product of Bessel functions, F can be written exactly as the following integral of the Riemann function:

$$F(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)e^{ik\tau}R(\xi=\tau,\eta=\tau;s,p)\,d\tau \qquad (5.2a)$$

$$= \frac{H(s)}{m}\int_0^s V(\tau)e^{ik\tau}J_0\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right)d\tau. \qquad (5.2b)$$

This expression is, of course, the well-known flat-plate solution obtainable by a number of other methods (e.g. Miles 1959, p. 50).

Motivated by (5.2a), in the present case of a parabolic airfoil we try

$$F(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)e^{ik\tau}R(\xi=\tau,\eta=\tau;s,p)\,d\tau, \qquad (5.3)$$

where R is now given by (4.8) and this can be directly verified to satisfy the governing equation (3.8a). Also, substituting this into the boundary condition (3.8b) and recalling that it is valid to $O(\theta_0)$, it can be shown by using some of the results obtained by the present author (1974) that the boundary condition is indeed satisfied to the same order, the details being given in appendix A. From (5.3), F obviously vanishes for $s < 0$. Hence (3.8c) is satisfied and (5.3) is in fact the solution sought. Before we write down the final solution explicitly, we restore, in order to obtain $\psi^{(1)}$ in (3.7), the exponential factor, which may be written to the present order of approximation as

$$\exp\left[i\frac{\epsilon k}{U_\infty}N(p-s)\phi'^{(1)}(s)\right] \simeq \exp\left[i\frac{\epsilon k}{U_\infty}N2my\,\phi'^{(1)}(s)\right].$$

When we collect all the results obtained so far, we have the following: if the airfoil shape in the mean position is given by

$$\epsilon f(x) = \epsilon(\tfrac{1}{2}\alpha x^2+\beta x),$$

where $\alpha < 0$ (a convex surface), and the co-ordinate of the moving upper surface is given by

$$y = \epsilon f(x) - \theta_0 e^{i\omega t}g(x),$$

where the amplitude of the motion $g(x)$ is an arbitrary function of x, then the leading term of the unsteady part of the velocity potential, $\psi^{(1)}$ in (3.2b), becomes

$$\begin{aligned}\psi^{(1)}(s,p) = \frac{H(s)}{m}\int_0^s & V(\tau)\exp(ik\tau)\exp\left[i\frac{\epsilon k}{U_\infty}N2my\,\phi'^{(1)}(s)\right]\\ &\times\exp\left\{i\frac{\epsilon k}{U_\infty}[(N-1)(\phi^{(1)}(s)-\phi^{(1)}(\tau))+N(s-\tau)\phi'^{(1)}(s)]\right\}\\ &\times M\left[\frac{1}{2}-\frac{mk}{4i\epsilon\alpha NM_\infty^2},1,\frac{i\epsilon k\alpha N}{m}(s-\tau)(p-\tau)\right]d\tau, \qquad (5.4)\end{aligned}$$

where

$$V(x) = U_\infty g'(x) + \frac{ikU_\infty m^2}{M_\infty^2} g(x), \quad k = \frac{\omega}{U_\infty}\left(\frac{M_\infty}{m}\right)^2, \quad N = \frac{\gamma+1}{2}\left(\frac{M_\infty}{m}\right)^2,$$

$$\phi^{(1)}(s) = (-U_\infty/m)(\tfrac{1}{2}\alpha s^2 + \beta s),$$

$$s = \frac{x - my + M_\infty^2 N\epsilon\beta y}{1 - M_\infty^2 N\epsilon\alpha y}, \quad p = x + my - \frac{\epsilon}{2U_\infty}(N-2)M_\infty^2[\phi^{(1)}(s) - \phi^{(1)}(p)],$$

$$|(\epsilon/m)(\alpha s + \beta)| \ll 1.$$

This integral representation is the solution we have been seeking.† (The last inequality is a restriction due to the assumption of a small perturbation.) Before attempting to extract physical meanings, we pause in the next section to observe that the present solution embraces the various known results as special limiting cases.

6. Limiting cases

6.1. *Steady limit*

In the limit $\omega \to 0$ or $k \to 0$, from the limiting form $M(a, b, 0) = 1$ of the confluent hypergeometric function (e.g. Abramowitz & Stegun 1964, p. 108), (5.4) is immediately reduced to

$$\psi^{(1)} = \frac{H(s)}{m}\int_0^s V(\tau)\,d\tau$$

$$= \frac{H(s)}{m} U_\infty g(s), \tag{6.1}$$

from (3.6*b*). This is Whitham's rule for steady flow and becomes identical to (3.4) if we replace f by $-g$. We wish to emphasize that g is an arbitrary function and that we have recovered the above as the limit for zero frequency of oscillation.

6.2. *Oscillating flat-plate airfoil*

In the limit $\epsilon \to 0$, when we note that (Abramowitz & Stegun 1964, p. 506)

$$\lim_{a\to\infty} M(a, 1, -z/a) = J_0(2z^{\frac{1}{2}}),$$

(5.4) becomes at once

$$\psi^{(1)} = \frac{H(x-my)}{m}\int_0^{x-my} V(\tau)\,e^{ik\tau} J_0\left(\frac{k}{M_\infty}[(x-\tau)^2 - m^2y^2]^{\frac{1}{2}}\right)d\tau, \tag{6.2}$$

which is precisely the well-known flat-plate solution; the physical meaning of this integral representation was given by the present author (1974).

† In this connexion, it is of interest to note that Goldstein & Rice (1973) found a solution for sound propagating through a uniform shear flow in terms of the parabolic cylinder function, which is intimately connected with the confluent hypergeometric function.

6.3. *Oscillating wedge*

The third case which invites comparison with the present result is that of an oscillating wedge. In the limit $\alpha \to 0$, with the aid of the limiting formula cited in §6.2 we obtain

$$\psi^{(1)} = \frac{H(s)}{m}\int_0^s V(\tau)\, e^{ik\tau} \exp(-2iN\epsilon\beta ky) \exp\left[-\frac{i}{m}\,\epsilon\beta k(2N-1)(s-\tau)\right] \times J_0\left(\frac{k}{M_\infty}\,[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right) d\tau, \tag{6.3}$$

where

$$s = x - my + M_\infty^2 N\epsilon\beta y,$$

$$p = x + my - (N-2)\, M_\infty^2\, \epsilon\beta y$$

and $\epsilon\beta$ is the semi-vertex angle of the wedge.

In order to confirm the agreement of this formula with that obtained by previous workers, we first restore the factor e^{-ikx} to (6.3). It is convenient to rotate the co-ordinate system from (x, y) to (x_2, y_2), where x_2 is parallel to the upper surface of the wedge and y_2 normal to it. At the same time, we refer the flow properties to the mean steady flow behind the shock instead of those upstream of the shock and designate them by a subscript 2. Furthermore, we change the integration variable from τ to $\eta \equiv \tau(1 + m_2\epsilon\beta)$. All this transforms the right-hand side of (6.3), upon discarding negligible quantities, into the following expression:

$$\theta_0 e^{-ikx}\psi^{(1)} \sim \theta_0 \frac{1}{m_2}\, H(x_2 - m_2 y_2) \int_0^{x_2 - m_2 y_2} V(\eta) \exp(-ik_2 x_2) \exp(ik_2\eta) \times J_0\left(\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right) d\eta. \tag{6.4}$$

This is identical to the flat-plate solution (6.2) if the latter is expressed in terms of the (x_2, y_2) co-ordinate system and the flow properties downstream of the shock. This result is not unexpected, since it is known that, if one takes the second-order equation for the unsteady component of the velocity potential to $O(\epsilon\theta_0)$ and expresses it in terms of these co-ordinate systems and flow variables, then for a wedge it exactly reduces to the acoustic equation. (The reason why the relationship (6.4) is approximate rather than exact is obviously due to the fact that, in the course of applying the strained co-ordinate technique, some non-essential second-order terms have been discarded.)

Carrier (1949) obtained a solution for a wedge oscillating at its apex; the solution was derived in a more generalized way by including the rippling motion of the shock and, in addition to the irrotational component of the flow, rotational flow behind the shock. His solution was later generalized to include the case of a moving vertex by Van Dyke (1953*b*), who also corrected typographical errors in Carrier's paper. The solution was expressed in the form of a series involving Bessel functions. In order to facilitate direct comparison, we recast the present solution (6.4) in the following alternative form:

$$\theta_0 \frac{1}{m_2}\int_0^{x_2 - m_2 y_2} V(\eta) \exp[-ik_2(x_2-\eta)]\, J_0\left[\frac{k_2}{M_2}\,[(x_2-\eta)^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right] d\eta = -\theta_0 a_2 \sum_{\nu=1}^{\infty} b_\nu e^{-\nu\theta} J_\nu \left[\frac{k_2}{M_2}\,[x_2^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right] \exp(-ik_2 x_2), \tag{6.5a}$$

where

$$\tanh\theta = m_2 y_2/x_2,$$
$$b_\nu = (iM_2\nu/k_2 m_2)[t^\nu + (-t)^{-\nu}] + b\cos(\epsilon\beta)[t^\nu - (-t)^{-\nu}], \tag{6.5b}$$
$$t = i(M_2 + m_2)$$

and where $V(x_2) = U_2 + i\omega(x_2 - b\cos\epsilon\beta)$, b being the pivotal position of the oscillating wedge measured from the apex. The above identity is given in appendix B. Now Carrier's solution for the irrotational component of the flow becomes, in the present notation,

$$\theta_0 a_2 \sum_{\nu=1}^{\infty} [a_\nu \cosh\nu\theta + b_\nu \sinh\nu\theta]\, J_\nu\left[\frac{k_2}{M_2}[x_2^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right]\exp(-ik_2 x_2). \tag{6.6}$$

(The expression for b_ν given in (6.5b) is the corrected one given by Van Dyke 1953b.) Carrier showed that as long as the shock is sufficiently weak

$$a_\nu \doteqdot -b_\nu,\dagger$$

and in such a case (6.6) is indeed identical to the right-hand side of (6.5a). This agreement naturally endorses the present viewpoint that the global behaviour of the unsteady flow downstream of the weak bow shock can be determined essentially independently of the presence and movement of the shock.

7. Alternative representation of the solution and interpretation

Returning now to the immediate subject of a parabolically curved airfoil, the solution as given in (5.4) is not appropriate for extracting its physical significance. Such an interpretation will, however, be obvious once we recast (5.4) in a more revealing form by making use of the following Tricomi (1949) expansion formula for the confluent hypergeometric function in a series of Bessel functions:

$$M(a,b,x) = \Gamma(b)(\lambda x)^{\frac{1}{2}(1-b)}\exp(\tfrac{1}{2}x)\sum_{n=0}^{\infty} A_n\left(\frac{x}{4\lambda}\right)^{\frac{1}{2}n} J_{n+b-1}[2(\lambda x)^{\frac{1}{2}}] \quad \text{for} \quad \operatorname{Re} b > 0, \tag{7.1}$$

where λ is the Whittaker parameter, given by $\lambda = \frac{1}{2} - a$, and

$$A_0 = 1, \quad A_1 = 0, \quad A_2 = \tfrac{1}{2},$$
$$(n+2)A_{n+2} = (n+1)A_n - 2\lambda A_{n-1}.$$

When we insert this into (5.4), we obtain

$$\begin{aligned}\psi^{(1)}(s,p) = \frac{H(s)}{m}\int_0^s V(\tau)\,e^{ik\tau}\exp(-i\epsilon k\sigma) &\\ \times\Bigg\{J_0\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right) + \sum_{n=2}^{\infty} A_n\left[-\left(\frac{N\epsilon\alpha M_\infty}{m}\right)^2(s-\tau)(p-\tau)\right]^{\frac{1}{2}n} &\\ \times J_n\left(\frac{k}{M_\infty}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right)\Bigg\}\,d\tau,&\end{aligned} \tag{7.2a}$$

† Van Dyke (1953b, also private communication) proved that for a small wedge angle

$$a_1/b_1 = -1 + 2ib\theta_0\epsilon/b_1 + O(\epsilon^2).$$

As for the rotational component of the flow, the first term of its series representation may be shown to be $O(\theta_0\epsilon^{\frac{3}{2}})$.

where

$$A_2 = \tfrac{1}{2}, \quad A_3 = -\tfrac{2}{3}\lambda, \quad A_4 = \tfrac{3}{8}, \tag{7.2b}$$

$$(n+1)A_{n+1} = nA_{n-1} - 2\lambda A_{n-2}, \quad \lambda = mk(4iM_\infty^2 N\epsilon\alpha)^{-1},$$

$$\sigma = -2Nmy\frac{1}{U_\infty}\phi'^{(1)}(s) - (N-1)\frac{1}{U_\infty}[\phi^{(1)}(s)-\phi^{(1)}(\tau)] - N(s-\tau)\frac{1}{U_\infty}\phi'^{(1)}(s) - \frac{\alpha N}{2m}(s-\tau)(p-\tau)$$

$$\simeq -\frac{Nmy}{U_\infty}[\phi'^{(1)}(s)+\phi'^{(1)}(\tau)] - \frac{1}{U_\infty}(2N-1)[\phi^{(1)}(s)-\phi^{(1)}(\tau)].$$

(All three of the limiting cases of the preceding section are now directly derivable from the present form: for example, when k is set equal to zero, the result (6.1) follows at once.) Equation (7.2a) immediately surrenders itself to the following physical interpretation. Let us first examine the flow field near the leading edge, where both y and s are small. Then (7.2a) becomes, approximately,

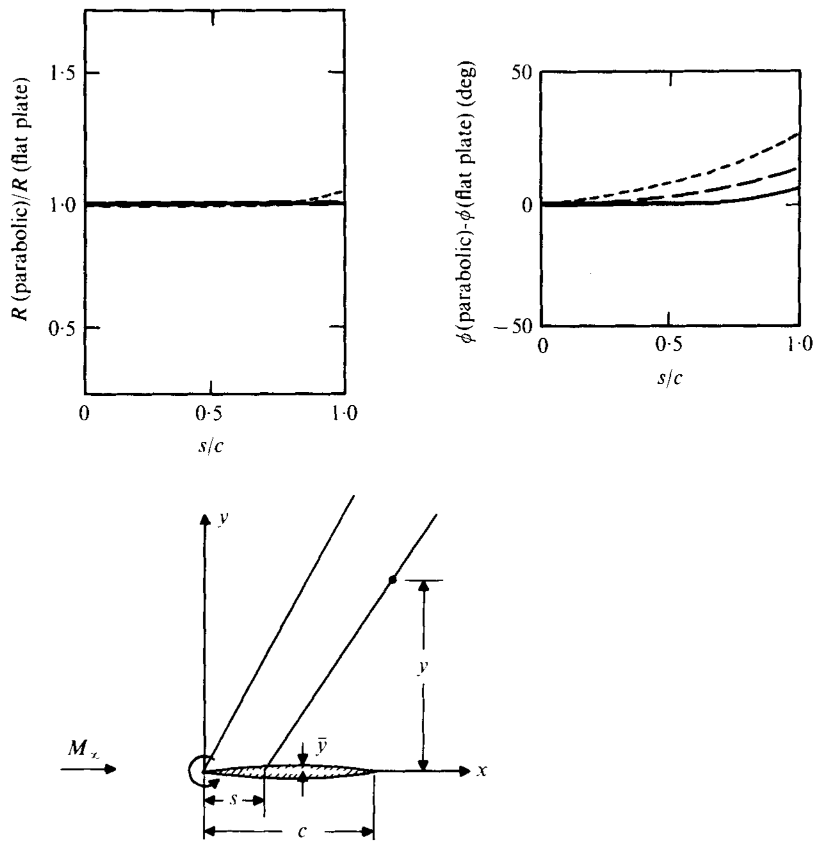
$$\psi^{(1)} = \frac{H(x-my)}{m}\int_0^{x-my} V(\tau)\,e^{ik\tau}J_0\left(\frac{k}{M_\infty}[(x-\tau)^2-(my)^2]^{\frac{1}{2}}\right)d\tau.$$

This is the flat-plate solution (6.2), and in this region the effect of the body shape is indiscernible as yet; the unsteady flow field is completely separated from the non-uniform, steady flow. Physically the decoupling occurs because the unsteady disturbance, having travelled only a short distance from the leading edge, has suffered little distortion.

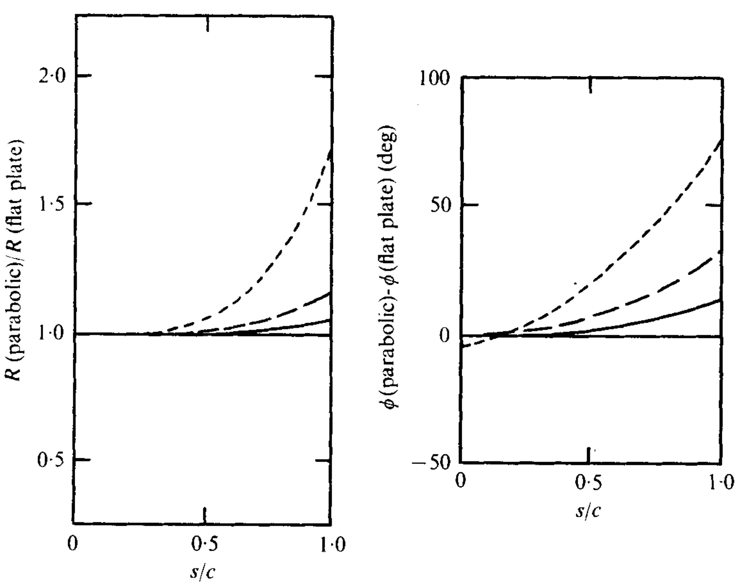
We now move away from the leading edge by increasing the value of y while keeping the value of s constant (along the straight Mach wave) or penetrate downstream by increasing the value of s while keeping y constant. In either case, if we look at the integrand of (7.2a) or the signal emitted at a point τ on the airfoil, the complex exponential term, which can be written as

$$\exp(-i\epsilon k\sigma) = \exp\left\{\frac{i\epsilon k}{U_\infty}[Nmy(\phi'^{(1)}(s)+\phi'^{(1)}(\tau)) + (2N-1)(\phi^{(1)}(s)-\phi^{(1)}(\tau))]\right\},$$

immediately discloses the following key aspect: no matter how slender ($\epsilon \ll 1$) the airfoil may be, this phase shift (induced by the presence of the body) will eventually amount to an increasing delay at a large distance y or s. Moreover, it is also crucial to recognize here that the phase lag of the signal received at a position s depends not only on the local flow at that point, but also, through the very difference in the steady velocity potential, i.e. $\phi^{(1)}(s)-\phi^{(1)}(\tau)$, upon the entire flow field which the signal has traversed; the disturbance 'remembers' its past. Thus we might call this exponential factor the phase memory, a term commonly used in connexion with the propagation of a radio wave through a stratified ionosphere (e.g. Budden 1961). As stated in the introduction, the existence of phase memory, which differs from one signal to another, is by itself quite sufficient to induce, upon superposition, a change in the amplitude of the unsteady flow field. The change is, however, further enhanced because the shape of the airfoil alters even the amplitude of the individual signal in the far field when the contributions from the higher-order terms of (7.2a) in the series of Bessel functions begin to surface. Thus, in the far field the airfoil shape, in its effect of causing non-

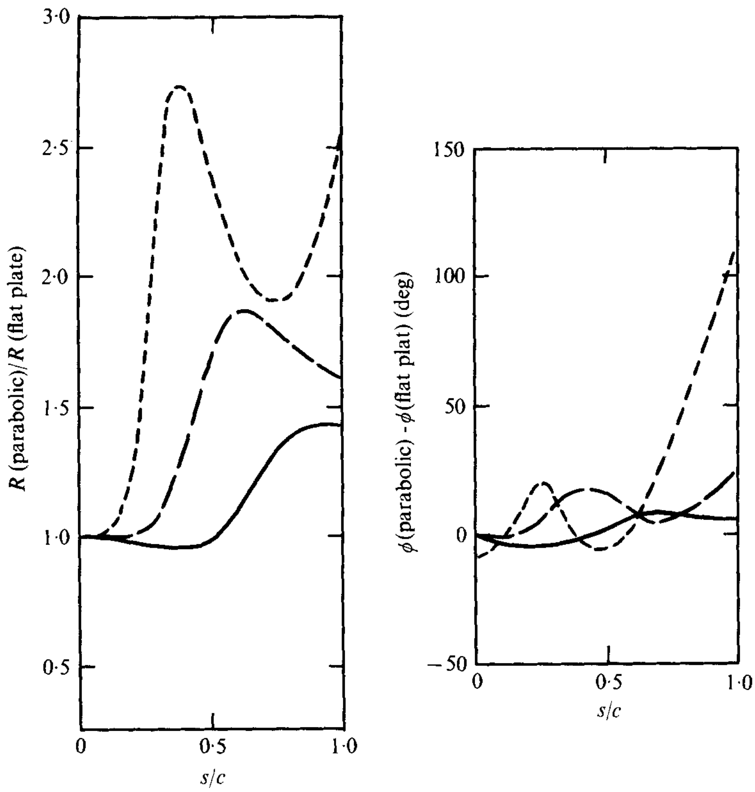


(*a*)



(*b*)

FIGURES 4 (*a*, *b*). For legend see next page.



(c)

FIGURE 4. Amplitude and phase of unsteady pressure; $-p'(\rho_\infty U_\infty^2 e^{i\omega t})^{-1} = Re^{i\phi}$. The ordinate for the left figure of each pair is the ratio of the amplitude R for a parabolic-arc airfoil to that for a flat plate. The ordinate on the right is the difference in phase ϕ; $M_\infty = 1{\cdot}3$, $\gamma = 1{\cdot}4$, $\epsilon = 0{\cdot}1$, $\alpha = -1$, $\beta = 0{\cdot}5$ (max $\bar{y}/c = 0{\cdot}0125$) and the pivot axis is at the leading edge. $\bar{y}/c = \epsilon[\frac{1}{2}\alpha(x/c)^2 + \beta(x/c)]$. (a) $\omega c/U_\infty = 0{\cdot}1$ ($kc = 0{\cdot}245$): ——, $y/c = 0$ ($ky = 0$); – –, $y/c = 0{\cdot}82$ ($ky = 0{\cdot}2$); ----, $y/c = 2{\cdot}04$ ($ky = 0{\cdot}5$). (b) $\omega c/U_\infty = 0{\cdot}3$ ($kc = 0{\cdot}735$): ——, $y/c = 0$ ($ky = 0$); – –, $y/c = 0{\cdot}82$ ($ky = 0{\cdot}6$); ----, $y/c = 2{\cdot}04$ ($ky = 1{\cdot}5$). (c) $\omega c/U_\infty = 1$ ($kc = 2{\cdot}45$): ——, $y/c = 0$ ($ky = 0$): – –, $y/c = 0{\cdot}82$ ($ky = 2$); ----, $y/c = 2{\cdot}04$ ($ky = 5$).

uniform surrounding flow, is inextricable from the unsteady flow field and deeply affects both its phase and amplitude, as well as the directional change in the characteristic curves.

This point is illustrated in figure 4, where the unsteady pressure distribution for a parabolic airfoil (max $\bar{y}/c = 0{\cdot}0125$), computed from (7.2),† is compared with the result for a flat-plate airfoil at three different frequencies of oscillation: $\omega c/U_\infty = 0{\cdot}1$ in figure 4(*a*), $\omega c/U_\infty = 0{\cdot}3$ in figure 4(*b*) and $\omega c/U_\infty = 1$ in figure 4(*c*). There, both the amplitude R and phase ϕ are plotted as functions of s, i.e. the distance between the root of a straight Mach wave and the leading edge, and at three different values of y. (If the flow were steady then, regardless of y, the amplitude would remain the same along the characteristics s = constant.) We observe that, though for $\omega c/U_\infty = 0{\cdot}1$ the effect of the airfoil shape does not become prominent at these values of y, it begins

† For numerical computations, (7.2) is also more convenient than (5.4).

to emerge at $\omega c/U_\infty = 0{\cdot}3$; and for $\omega c/U_\infty = 1$, except for the close vicinity of the leading edge, it indeed alters the pressure distribution substantially.

The profound modification of the unsteady linear theory displayed here raises an obviously disquieting thought on the upshot of the acoustic theory when multi-body aerodynamic interference is involved and deepens concern expressed (Kurosaka 1975) with regard to some of the consequences arising from a *pro forma* sum of linearized unsteady upwashes.

8. Further interpretation

Pursuing the physical interpretation further, we seek the connexion between (7.2a) and the 'fundamental' solution. We shall not, however, merely reconstruct (7.2a) by the superposition of the fundamental solution. Rather, we shall reverse the usual process and obtain the fundamental solution from (7.2a): that is to say, we regard (7.2) as the spectrum at frequency ω or the Fourier transform and take its inverse transform so as to derive the transient response to an arbitrary time-dependent motion of the airfoil. The 'fundamental' solution will arise naturally in the course of obtaining the transient response (Miles 1959, p. 53). Let us go back to (2.4) and rewrite the unsteady part in a more general way as

$$\Phi = \epsilon\phi + \theta_0\,\Omega(x, y:t).$$

Then the Fourier transform $\tilde{\Omega}(\omega)$ of Ω (its leading part) is equal to $e^{-ikx}\psi^{(1)}$, $\psi^{(1)}$ being given by (7.2a), provided that V is regarded as the Fourier transform $\tilde{V}$ of itself, i.e.

$$\begin{aligned}\tilde{\Omega}(\omega) = \frac{1}{m}\int_0^s \tilde{V}(\tau)\exp\left\{-i\frac{\omega M_\infty}{a_\infty m^2}[(x-\tau)+\epsilon\sigma(\tau)]\right\} \\ \times\left\{J_0\left[\frac{\omega}{a_\infty m^2}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right]\right. \\ \left.+\sum_{n=2}^{\infty} A_n(\omega)\,c_n J_n\left[\frac{\omega}{a_\infty m^2}[(s-\tau)(p-\tau)]^{\frac{1}{2}}\right]\right\}d\tau,\end{aligned}$$

where
$$c_n = \left[-\left(\frac{N\epsilon\alpha M_\infty}{m}\right)^2(s-\tau)(p-\tau)\right]^{\frac{1}{2}n}.$$

Taking the inverse transform

$$\Omega = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\tilde{\Omega}\,e^{i\omega t}\,d\omega,$$

we obtain, by convolution,

$$\Omega(x,y,t) = \frac{1}{m}\int_0^s d\tau\int_{-\infty}^{\infty} V(\tau, t-\xi)F(\xi)\,d\xi. \tag{8.1a}$$

Here

$$[\partial\Omega/\partial y]_{y=0} = V(x,t)$$

and
$$F(\xi) = \frac{1}{\pi}\sum_{n=0}^{\infty} c_n F_n, \tag{8.1b}$$

where, for example,

$$F_0 = r^{-1}H(b-|a|),\quad F_1 = 0,\quad F_2 = -(2r)^{-1}H(b-|a|)\cos 2\theta,$$

$$F_3 = \frac{i}{12N\epsilon\alpha M_\infty a_\infty mb}H(b-|a|)\left\{-\frac{8}{r}\cos 2\theta - \frac{2b^2}{r^3} + \ldots\right\}$$

and where

$$a = \xi - \frac{M_\infty}{a_\infty m^2}(x-\tau+\epsilon\sigma), \quad b = \frac{1}{a_\infty m^2}[(s-\tau)(p-\tau)]^{\frac{1}{2}},$$
$$\cos\theta = a/b, \quad r = (b^2-a^2)^{\frac{1}{2}}.$$

We note that $\Omega(x,y,t)$ contains, through F_n in (8.1*b*), the term $1/r$, which can be written as

$$\frac{1}{r} = a_\infty m \Bigg/ \left\{\frac{1}{m^2}(s-\tau)(p-\tau) - \left[a_\infty m\xi - \frac{M_\infty}{m}(x-\tau+\epsilon\sigma)\right]^2\right\}^{\frac{1}{2}}. \tag{8.2}$$

The meaning will become immediately recognizable if we note that at $\epsilon = 0$ the denominator of (8.2) may be reduced, after some algebra, to

$$\{(a_\infty\xi)^2 - (x-\tau-U_\infty\xi)^2 - y^2\}^{\frac{1}{2}}.$$

This represents, when set equal to zero, a circular wave front of a disturbance which was emitted at a source point $(\tau, 0)$ and is propagating through uniform flow after a time ξ. Thus the denominator of (8.2), when put equal to zero, i.e.

$$\frac{1}{m^2}(s-\tau)(p-\tau) - \left[a_\infty m\xi - \frac{M_\infty}{m}(x-\tau+\epsilon\sigma)\right]^2 = 0, \tag{8.3}$$

now describes the *distorted* wave front propagating in a non-uniform flow field. In fact, we can directly show that the expression for ξ obtained from (8.3) does satisfy, within the approximation consistent with the present analysis, the appropriate eikonal equation at large distances; (8.2) is indeed the fundamental solution. In general, for a given point (x,y) in flow and for a given source point $(\tau, 0)$, there are two values of ξ satisfying (8.3): one corresponds to the time when the disturbance first arrives at (x,y) and the other to the time when it departs from (x,y). In the particular case when the point (x,y) is located such that either

$$s = \tau \quad \text{or} \quad p = \tau,$$

there is only one such moment for ξ, which implies that the wave front is tangential to either $s = \tau$ or $p = \tau$. $s = \tau$ corresponds to the straight Mach wave, whose root is located at $(\tau, 0)$; $p = \tau$ is the cross Mach wave passing through the same point. Hence, as expected, two families of Mach waves passing the source point form envelopes for the disturbance emitted from the source. In particular, the time required for the signal to arrive at a point on the straight Mach line $s = \tau$ is given by

$$\xi = \frac{M_\infty y}{a_\infty m}\left[1 + \epsilon(\alpha s+\beta) N \frac{(2-M_\infty^2)}{m}\right]. \tag{8.4}$$

It is of interest to note that this can be obtained in the following, more physical way. The wave-front velocity $\mathbf{c}$ is in general the vectorial sum of the local acoustic speed in the direction of the normal $\mathbf{n}$ to the front and the convective fluid velocity, i.e. $\mathbf{c} = a\mathbf{n} + \mathbf{u}$. However, along the enveloping Mach waves, which are tangential to the wave front, the acoustic speed does not contribute to the component of the wave-front velocity in the direction parallel to the Mach wave; only the fluid velocity contributes. In particular, along the straight Mach line the component of the fluid velocity or the wave-front velocity remains constant. If we divide the distance from the source $(\tau, 0)$ to the point (x, y) by the component of the flow velocity in the direction of the straight Mach wave, we can directly derive (8.4), as the time elapsed.

9. Concluding remarks

It has been our aim to find a uniformly valid solution for the unsteady flow field and examine it in detail. We have shown, through an explicit solution obtained for the specific case of a parabolic-arc airfoil oscillating in supersonic flow, that the prediction of the unsteady signal in the far field demands the detailed description of the contour of the moving boundary. The non-uniform surrounding flow produced by the very presence of the body, no matter how slender it may be, cumulatively and inextricably affects both the amplitude and the phase of the unsteady disturbance at a large distance from the leading edge.

As a further related effort, it would appear to be worth while to pursue a study for other airfoil shapes so as to enlarge our stock of particular solutions. With regard to the question of similar cumulative, first-order effects of nonlinearity in subsonic flow, we still remain uncertain. It is intriguing, however, to note that in a very recent paper of Goldstein & Atassi (1976), where an exact second-order solution is obtained for an airfoil subject to a convected gust, the incoming gust, in its nonlinear interaction with the steady non-uniform flow field, is found to suffer distortion in wavelength in a manner akin to the present supersonic result though the flow treated there is incompressible.

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Appendix A

In this appendix we shall show that the expression for F given by (5.3) does satisfy the boundary condition (3.8*b*) to order θ_0. We denote the left-hand side of (3.8*b*) by

$$l(F) \equiv \theta_0 \left\{ i\epsilon k \left[-\frac{1}{m} f'(s) - \frac{2}{U_\infty} N \phi'^{(1)}(s) \right] F + F_s - F_p - \frac{1}{m} V(s) e^{iks} \right\}, \tag{A 1}$$

and we shall prove that along $s = p$ this vanishes, to order θ_0. Substituting (5.3) into the above, one obtains for $s > 0$

$$l(F) = \frac{\theta_0}{m} \int_0^s V(\tau) e^{ik\tau} \exp\left(i \frac{\epsilon k}{U_\infty} \nu \right) \times \left\{ i\epsilon k \left[-\frac{1}{m} f'(s) - \frac{2N}{U_\infty} \phi'^{(1)} + \frac{1}{U_\infty} (\nu_s - \nu_p) \right] M + (M_s - M_p) \right\} d\tau, \tag{A 2}$$

where

$$\nu = (N-1)[\phi^{(1)}(s) - \phi^{(1)}(\tau)] - N(\tau - s)\phi'^{(1)}(s),$$

$$M = M\left[\frac{1}{2} + \frac{U_\infty k}{4i\epsilon N M_\infty^2 \phi''^{(1)}},\ 1,\ -\frac{i}{U_\infty} \epsilon k N (s-\tau)^2 \phi''^{(1)} \right].$$

We observe that along $s = p$

$$\nu_s = (2N-1)\phi'^{(1)}(s) - N(\tau - s)\phi''^{(1)}, \quad \nu_p = 0, \quad M_s = M_p. \tag{A 3}$$

Furthermore, by Tricomi's expansion formula cited in §7, the confluent hypergeometric function M can be expressed as

$$M = \exp\left[-\frac{1}{2U_\infty} i\epsilon k N(s-\tau)^2 \phi''^{(1)}\right] \times \sum_{n=0}^{\infty} A_n \left[i\epsilon \frac{M_\infty}{U_\infty} N(s-\tau)\phi''^{(1)}\right]^n J_n\left[\frac{k}{M_\infty}(s-\tau)\right], \tag{A 4}$$

where

$$A_0 = 1, \quad A_1 = 0, \quad A_2 = \tfrac{1}{2}$$

and the other, higher-order A_n are the same as those given in (7.2b). The leading term of M is given by

$$M \sim \exp\left[-\frac{1}{2U_\infty} i\epsilon k N(s-\tau)^2 \phi''^{(1)}\right] J_0\left[\frac{k}{M_\infty}(s-\tau)\right]. \tag{A 5}$$

From (A 3) and (A 5), (A 2) becomes

$$l(F) = \theta_0 \epsilon k \left\{\frac{1}{m}\int_0^s V^*(\tau)\exp(iK\tau)\exp\left(i\frac{\epsilon k}{U_\infty}\nu\right)\exp\left[-\frac{i\epsilon k}{2U_\infty} N(s-\tau)^2\phi''^{(1)}\right] \times J_0\left[\frac{k}{M_\infty}(s-\tau)\right] d\tau\right\}, \tag{A 6}$$

where

$$V^*(\tau) = -V(\tau) i U_\infty^{-1} N(\tau - s)\phi''^{(1)}.$$

Equation (A 6) can contribute to $O(\theta_0)$ only when ϵk is such that, if properly nondimensionalized, $O(\epsilon k) = 1$ or $k = O(1/\epsilon)$. For such large values of k, we apply the following method of obtaining an asymptotic expansion (Kurosaka 1974): we first express J_0 in terms of an integral involving an exponential and use the stationary-phase method repeatedly. This yields

$$\int_0^s = O\left(\frac{1}{k}\right),$$

and (A 6) becomes

$$l(F) = O(\theta_0 \epsilon),$$

which is of higher order than $O(\theta_0)$; the other terms of (A 4) may similarly be shown to be of higher order. Hence to $O(\theta_0)$, $l(F) = 0$.

Appendix B

In this appendix we shall prove the identity (6.5a):

$$\theta_0 \frac{1}{m_2}\int_0^{x_2 - m_2 y_2} V(\eta)\exp(ik_2\eta)\exp(-ik_2 x_2) J_0\left(\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right) d\eta$$
$$= -a_2\theta_0 \sum_{\nu=1}^{\infty} b_\nu e^{-\nu\theta} J_\nu\left(\frac{k_2}{M_2}[x_2^2 - (m_2 y_2)^2]^{\frac{1}{2}}\right)\exp(-ik_2 x_2), \tag{B 1}$$

where

$$x_2 \geqslant m_2 y_2, \quad \tanh\theta = \frac{m_2 y_2}{x_2}, \quad b_\nu = \frac{iM_2\nu}{k_2 m_2}[t^\nu + (-t)^\nu] + b\cos(\epsilon\beta)[t^\nu - (-t)^{-\nu}],$$

$$t = i(M_2 + m_2), \quad V(x_2) = U_2 + i\omega(x_2 - b\cos\epsilon\beta). \tag{B 2}$$

First we write

$$V(\eta)\exp(ik_2\eta) = \frac{a_2}{\eta}\left[\left(M_2 - \frac{i\omega b}{a_2}\cos\epsilon\beta\right)[\eta\exp(ik_2\eta)]\right] + \left(i\frac{\omega}{a_2}\right)[\eta^2\exp(ik_2\eta)].$$

As suggested by Carrier (1949), we expand $\eta\exp(ik_2\eta)$ and $\eta^2\exp(ik_2\eta)$ in series of Bessel functions, through the generating function of the Bessel functions, and obtain

$$V(\eta)\exp(ik_2\eta) = -\frac{a_2}{\eta}\sum_{\nu=1}^{\infty} b_\nu \nu m_2 J_\nu\left(\frac{k_2}{M_2}\eta\right).$$

Substitution of this into the left-hand side of (B 1) yields

$$-a_2\theta_0 \sum_{\nu=1}^{\infty} b_\nu \nu \exp(-ik_2x_2) F(x_2), \tag{B 3}$$

where

$$F(x_2) = \int_0^{x_2}\left\{H[x_2 - m_2y_2 - \eta] J_0\left[\frac{k_2}{M_2}[(x_2-\eta)^2 - (m_2y_2)^2]^{\frac{1}{2}}\right]\right\}\left\{\frac{1}{\eta}J_\nu\left(\frac{k_2}{M_2}\eta\right)\right\}d\eta. \tag{B 4}$$

If we take the Laplace transform $\mathscr{F}$ of $F(x_2)$, defined by

$$\mathscr{F} = \int_0^\infty \exp(-sx_2)F(x_2)\,dx_2,$$

then, by convolution, we obtain

$$\mathscr{F} = \frac{1}{\nu}(k_2/M_2)^\nu\left[s^2 + \left(\frac{k_2}{M_2}\right)^2\right]^{-\frac{1}{2}}\left\{s + \left[s^2 + \left(\frac{k_2}{M_2}\right)^2\right]^{\frac{1}{2}}\right\}^\nu \exp\left\{-m_2y_2\left[s^2 + \left(\frac{k_2}{M_2}\right)^2\right]^{\frac{1}{2}}\right\}.$$

Inverting this gives (e.g. Erdélyi *et al.* 1954, p. 250)

$$F(x_2) = \frac{1}{\nu}e^{-\nu\theta}J_\nu\left(\frac{k_2}{M_2}[x_2^2 - (m_2y_2)^2]^{\frac{1}{2}}\right) \tag{B 5}$$

for $x_2 \geqslant m_2y_2$. By substituting (B 5) into (B 3), one may establish the required identity (B 1).

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